

Inflation Swap and Cap Valuation

A model is presented for pricing swaps, caps, and floors on inflation index returns. To capture general term structures of interest rates and index volatilities, the model requires time-averaged forward rate, and volatility inputs.

For a series of reset times

$$T_0 \leq 0 < T_1 < \dots < T_N = T,$$

we consider swap-type payments of the following form

- At maturity time T :
 - $\frac{I_T}{I_{T_0}} - 1$, (Zero swap)
 - $\max \left[0, \left(\frac{I_T}{I_{T_0}} - 1 \right) - X \right]$, (Zero cap)
 - $\max \left(0, X - \left(\frac{I_T}{I_0} - 1 \right) \right)$. (Zero floor)
- At each reset time T_k :
 - $\frac{I_{T_k}}{I_{T_{k-1}}} - 1$, (Reset swap)

- $\max \left[0, \left(\frac{I_{T_k}}{I_{T_{k-1}}} - 1 \right) - X \right]$, (Reset cap)
- $\max \left(0, X - \left(\frac{I_{T_k}}{I_{T_{k-1}}} - 1 \right) \right)$. (Reset floor)

where

- I_t denotes the level of the inflation index at time t ,
- X denotes the reset cap (floor) return strike.

We assume that the level of the inflation index satisfies a domestic risk-neutral SDE of the form

$$\frac{dI_t}{I_t} = (r(t) - \delta(t))dt + \sigma(t)dB_t \quad (1)$$

where

- r is the deterministic domestic continuously compounded risk-free,
- δ is a deterministic continuous dividend yield parameter,
- σ is a deterministic volatility parameter,
- B_t is a standard Brownian motion.

Given SDE (1) for index levels, we may model

$$\frac{I_T}{I_t} = F(0, t, T) \exp \left[-\frac{(\sigma_{t,T})^2 (T-t)}{2} + \sigma_{t,T} (W_T - W_t) \right], \quad (2)$$

where

- $F(0, t, T) = e^{\int_t^T (r(s) - \delta(s)) ds}$,

- $\sigma_{t,T} = \sqrt{\frac{1}{T-t} \int_t^T \sigma^2(s) ds}$,
- W_t is a standard Brownian motion.

The model

$$\frac{I_T}{I_t} = F(0,t,T) \exp \left[-\frac{(\sigma_{t,T}^{WM})^2 T}{2} + \sigma_{t,T}^{WM} W_T^{WM} \right], \quad (3)$$

where

- $F(0,t,T) = e^{\int_t^T (r(s) - \delta(s)) ds}$,
- $\sigma_{t,T}^{WM} = \frac{1}{\sqrt{T}} \sqrt{\int_t^T \sigma^2(s) ds}$,
- W_t^{WM} is a standard Brownian motion.

We note that, in marginal distribution, Equations 2 and 3 are equivalent.

In the inflation index market, forward levels $F(0,0,\tau_i)$ for a series of “key maturities” $\{\tau_i\}_{i=1}^m$ are traded. In particular, forward levels of the form

$$E_0^\tau \{I_\tau\} = \frac{1}{Z_0(\tau)} E_0^Q \left\{ e^{-\int_0^\tau r_s ds} I_\tau \right\},$$

where

- $Z_0(\tau)$ is the domestic discount factor, at time 0, to time τ ,

- r is the domestic continuously compounded risk-free rate.

Given the assumptions described in Sections 3.1 and 3.2 above, we have

$$E_0^\tau \{I_\tau\} = I_0 e^{\int_0^\tau (r(s) - \delta(s)) ds}$$

and

$$E_0^Q \left\{ \frac{I_{\tau+\Delta}}{I_\tau} - 1 \right\} = e^{\int_\tau^{\tau+\Delta} (r(s) - \delta(s)) ds} - 1 = \frac{F(0, 0, \tau + \Delta)}{F(0, 0, \tau)} - 1 \quad (4)$$

for all $\tau, \Delta > 0$.

References:

<https://finpricing.com/lib/FiBond.html>